Advanced Data Structures

Lecture 10: Temporal Data Structures 2

Florian Kurpicz

Organization

Exams

- 12.09.2023, 13.09.2023, and 21.09.2023 (09:00–16:00)
- 22.09.2023 (13:00–16:00)
- write to `blancani@kit.edu`
  - full name
  - Matrikelnummer
  - PO version
  - date
- in person
- 17.07.2022 Q&A during last half of lecture
- registration for project is open

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Evaluation

- next week

Range Minimum Queries in $O(1)$ Time and $O(n \log n)$ Space

- instead of storing all solutions
- store solutions for intervals of length 2^k for every k
- $M[0..n)[0..\lfloor \log n \rfloor)$

Queries

- query $rmq(A, s, e)$ is answered using two subqueries
- let $\ell = \lfloor log(e - s + 1) \rfloor$
- $m_1 = rmq(A, s, s + 2^\ell - 1)$ and $m_2 = rmq(A, e - 2^\ell + 1, e)$
- $rmq(A, s, e) = \arg\min_{m \in \{m_1, m_2\}} A[m]$

Construction

$$\begin{aligned} M[x][\ell] &= rmq(A, x, x + 2^\ell - 1) \\ &= \arg\min\{A[i] \colon i \in [x, x + 2^\ell)\} \\ &= \arg\min\{A[i] \colon i \in \{rmq(A, x, x + 2^{\ell-1} - 1), \\ &= \qquad rmq(A, x + 2^{\ell-1}, x + 2^\ell - 1)\}\} \\ &= \arg\min\{A[i] \colon i \in \{M[x][\ell - 1], \\ &= \qquad M[x + 2^{\ell-1}][\ell - 1]\}\} \end{aligned}$$

- dynamic programming in $O(n \log n)$ time

PINGO

https://pingo.scc.kit.edu/919720

Recap: Persistent Data Structures

- lecture based on: `http://courses.csail.mit.edu/6.851/spring12/lectures/L01`

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Definition: Partial Persistence

Only the latest version can be updated

Definition: Full Persistence

Any version can be updated

Definition: Confluent Persistence

Like full persistence, but two versions can be combined to a new version

Definition: Functional

Nodes cannot be modified, only new nodes can be created

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Retroactivity

- change in the past affects future
- make change in earlier version changes all later versions

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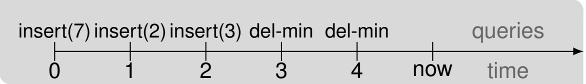
Retroactive Data Structures

Operations

- `INSERT`$(t, \mathit{operation})$: insert operation at time t
- `DELETE`(t): delete operation at time t
- `QUERY`(t, query): ask *query* at time t

- for a priority queue updates are
  - insert
  - delete-min
- time is integer ℹ for simplicity otherwise use order-maintenance data structure



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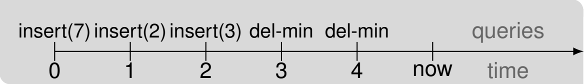

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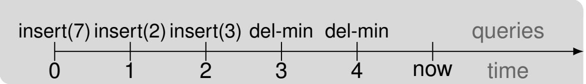

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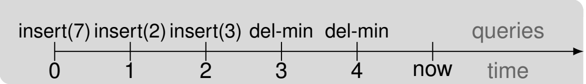

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Definition: Full Retroactivity

`QUERY` is allowed at any time t

Definition: Nonoblivious Retroactivity

`INSERT`, `DELETE`, and `QUERY` at any time t but also identify changed `QUERY` results

Easy Cases: Partial Retroactivity

- invertible updates
    - operation op^{-1} such that $op^{-}1(op(\cdot)) = \emptyset$
    - `DELETE` becomes `INSERT` inverse operation
- makes partial retroactivity easy
- $\texttt{INSERT}(t, operation) = \texttt{INSERT}(\infty, operation)$
- $\texttt{DELETE}(t, op) = \texttt{INSERT}(\infty, op^{-1})$

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Partial Retroactivity

- hashing
- dynamic dictionaries
- array with updates only ℹ $A[i] += value$

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Karlsruhe Institute of Technology

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- range minimum queries

- nearest neighbor
- point location
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- these types of problems are also “easy”

Decomposable Search Problems: Full Retroactivity

Lemma: Full Retroactivity for DSP

Every decomposable search problems can be made fully retroactive with a $O(\log m)$ overhead in space and time, where m is the number of operations

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Proof (Sketch)

- use balances search tree ⓘ segment tree
- each leaf corresponds to an update
- node n corresponds to interval of time $[s_n, e_n]$
- if an object exists in the time interval $[s, e]$, then it appears in node n if $[s_n, e_n] \subseteq [s, e]$ if none of n's ancestors' are $\subseteq [s, e]$
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- to query find leaf corresponding to t
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- data structure is stored for each operation!
- $O(\log m)$ space overhead!

General Full Retroactivity

Lemma: Lower Bound

Rewinding m operations has a lower bound of $\Omega(m)$ overhead

- general case

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Proof (Sketch)

- two values X and Y
- initially $X = \emptyset$ and $Y = \emptyset$
- supported operations
  - $X = x$
  - $Y += value$
  - $Y = X \cdot Y$
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Proof (Sketch, cnt.)

- perform operations
  - $Y+ = a_n$
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  - $Y+ = a_{n=1}$
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  - $\ldots$
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- what are we computing here? **PINGO**

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- $Y = a_n \cdot X^n + a_{n-1}X^{n-1} + \dots + a_0$
- evaluate polynomial at $X = x$ using `t=0,X=x`

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- evaluate polynomial at $X = x$ using `t=0,X=x`
- this requires $\Omega(n)$ time [FHM01]

Priority Queues: Partial Retroactivity (1/6)

- priority queue with
  - insert
  - delete-min
- delete-min makes PQ non-commutative

Lemma: Partial Retroactive PQ

A priority queue can be partial retroactive with only $O(\log n)$ overhead per partially retroactive operation

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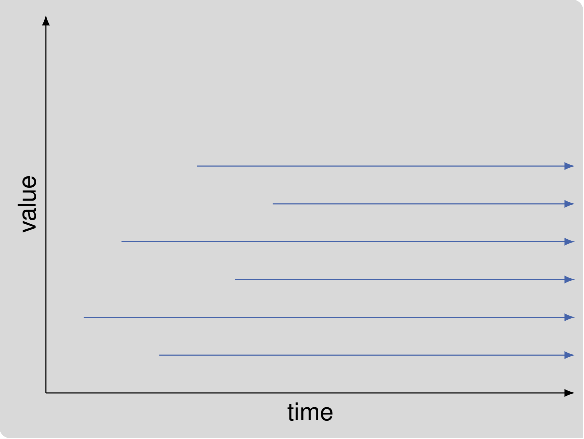

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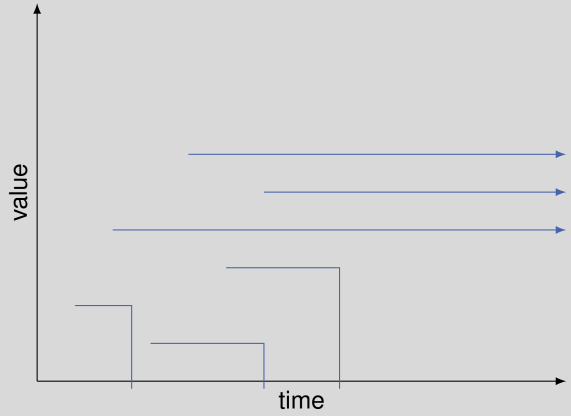

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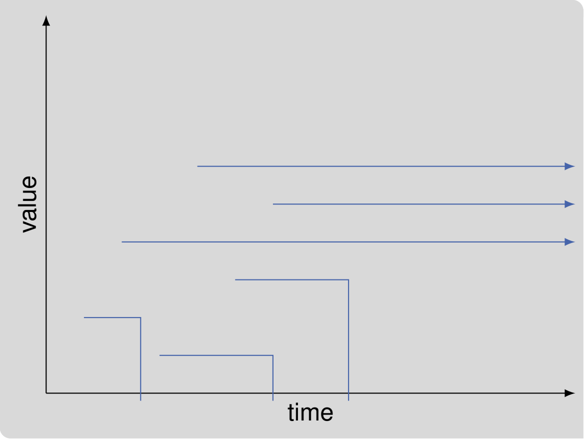

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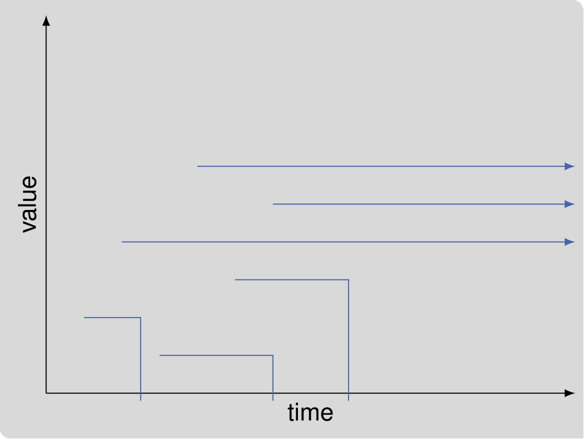

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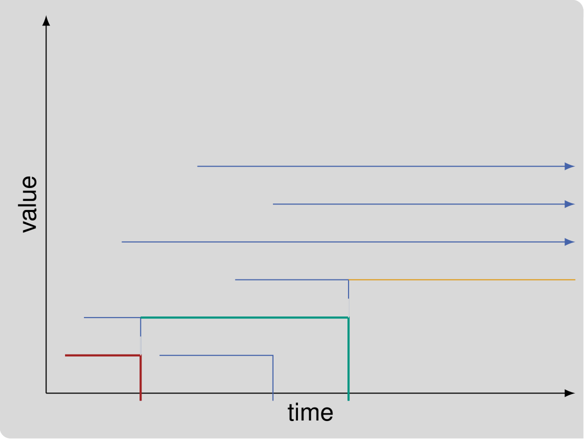

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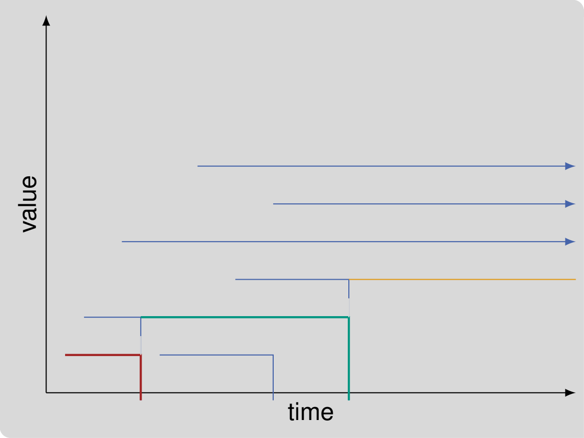

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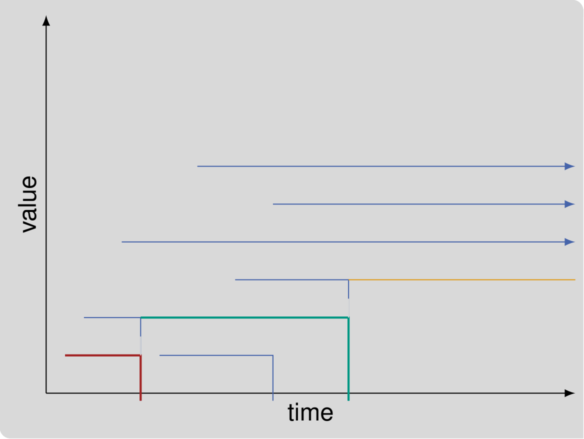
- insert deleted minimum right after deletion



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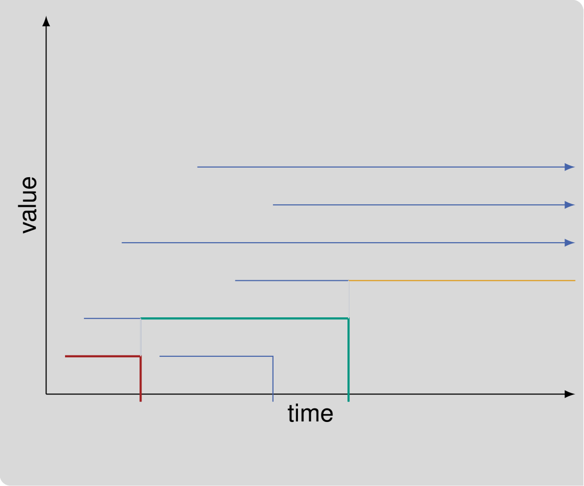
- let Q_t be elements in PQ at time t

- what values are in Q_∞? ℹ partial retroactivity
- what value inserts $\texttt{INSERT}(t, \text{insert}(v))$ in Q_∞
- values is $\max\{v, v' : v' \text{ deleted at time} \geq t\}$
- maintaining deleted elements is hard ℹ can change a lot



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A time t' is a bridge if $Q_{t'} \subseteq Q_\infty$

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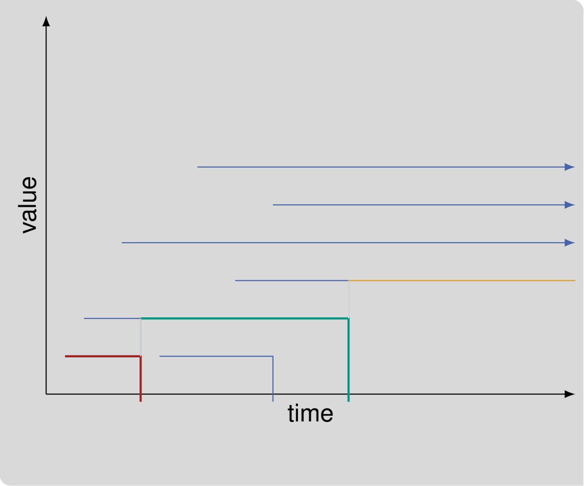

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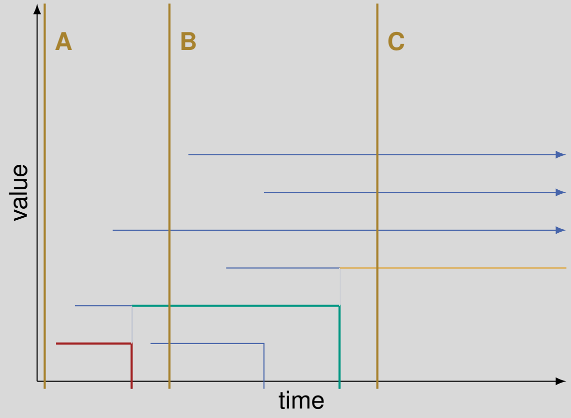

- what times are bridges? **PINGO**

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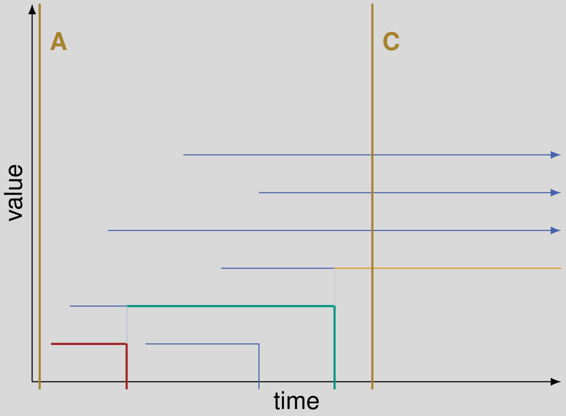

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If time t' is closest bridge preceding time t, then

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Proof (Sketch)

- $\max\{v' \notin Q_\infty : v' \text{ inserted at time} \geq t'\} \in \{v' : v' \text{ deleted at time} \geq t\}$
  - if maximum value is deleted between t' and t
  - then this time is a bridge
  - contradicting that t' is bridge preceding t

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Proof (Sketch, cnt.)

- $\max\{v' : v' \text{ deleted at time} \geq t\} \in \{v' \notin Q_\infty : v' \text{ inserted at time} \geq t'\}$
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  - then it is not in Q_∞

- what values are in Q_∞? ℹ partial retroactivity
- what value inserts $\texttt{INSERT}(t, \text{insert}(v))$ in Q_∞
- $\max\{v, v' \notin Q_\infty : v' \text{ inserted at time} \geq t'\}$

Priority Queues: Partial Retroactivity (5/6)

- keep track of inserted values
- use balanced binary search trees for $O(\log n)$ overhead

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- update all BBSTs in $O(\log n)$ time

Priority Queues: Partial Retroactivity (6/6)

Lemma: Partial Retroactive PQ

A priority queue can be partial retroactive with only $O(\log n)$ overhead per partially retroactive operation

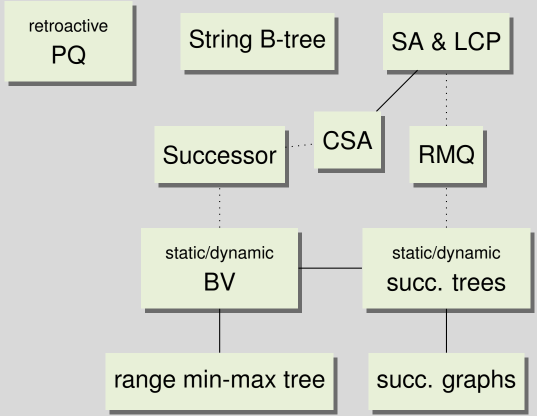
- requires three BBSTs
- updates need to update all BBSTs

Conclusion and Outlook

This Lecture

- retroactive data structures

Advanced Data Structures



Conclusion and Outlook

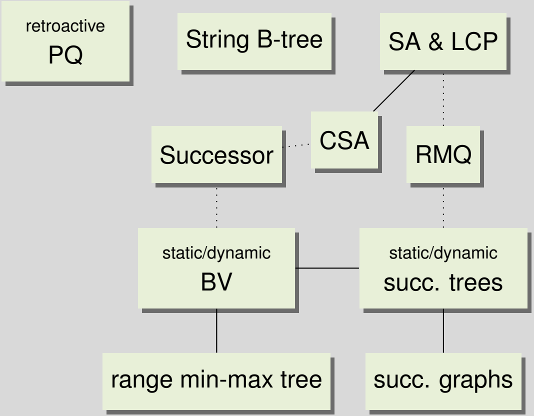
This Lecture

- retroactive data structures

Next Lecture

- (minimal) perfect hashing

Advanced Data Structures



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