Lecture 10: Lowest Common Ancestors

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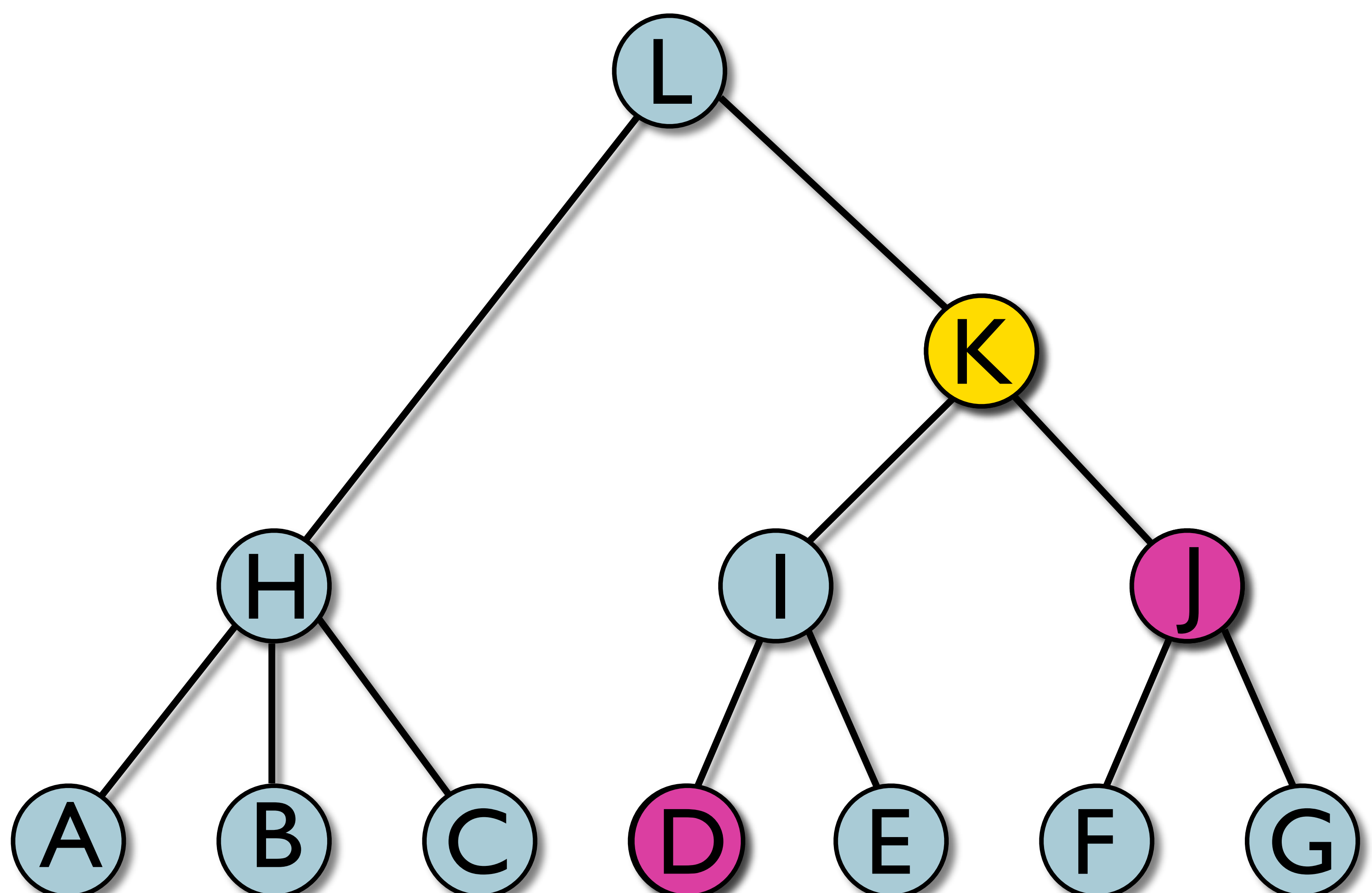
Lowest Common Ancestors



Some Initial Thoughts

- store only tree:

  $\Rightarrow$ $O(n)$ w.c. query time

- store all $\Theta(n^2)$ answers:

  $\Rightarrow$ $O(1)$ query time

- **difficulty**:
  - $O(1)$ query time with $O(n)$ space
  - lecture "Text Indexing" (SS'12 & SS'13)
  - here: **distributed** data structure

Distributed Data Structures

- no access to **global** data structures

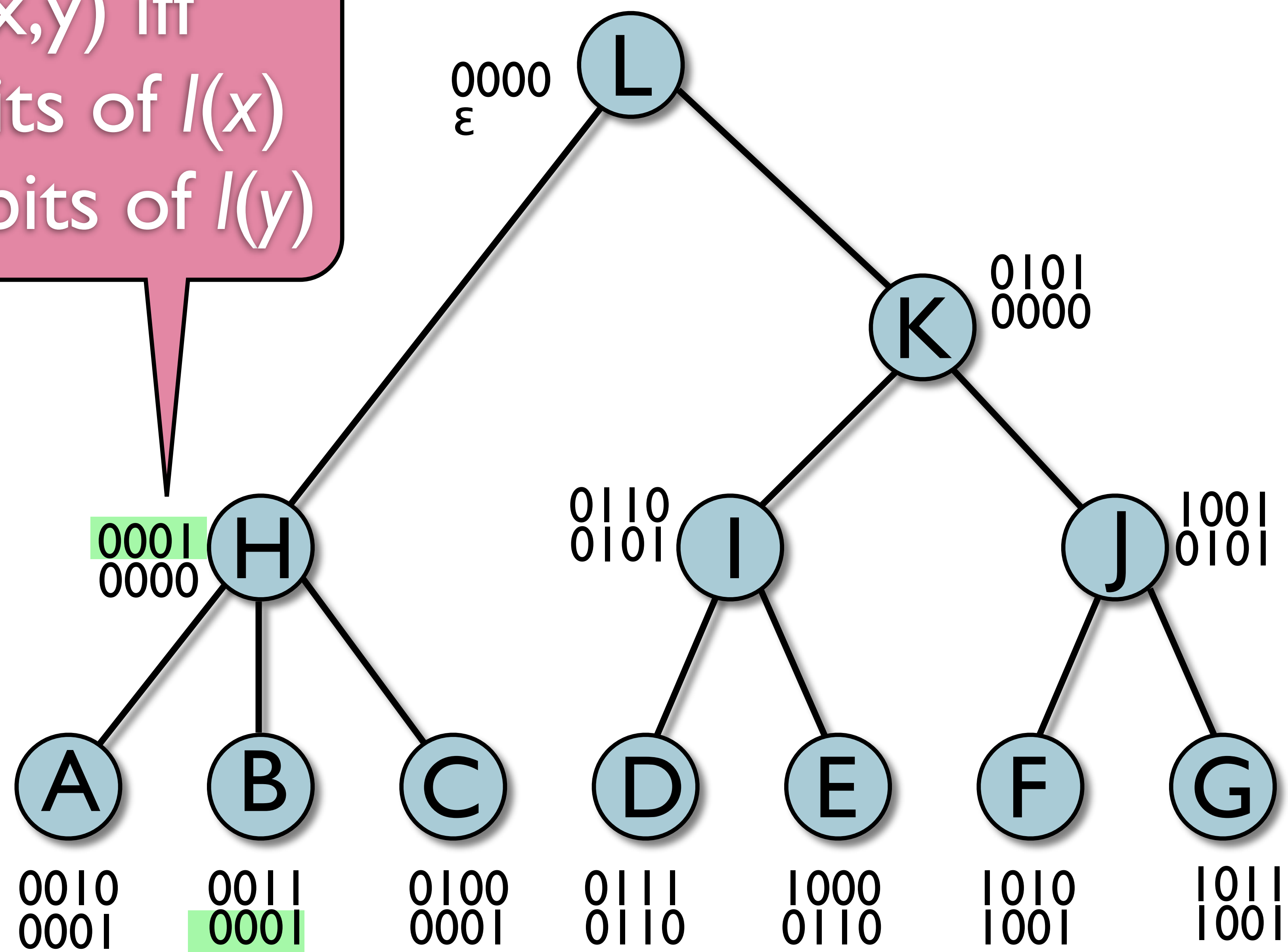
  $\rightsquigarrow$ minimize **communication overhead**

- **labeling scheme**:
  - assign label $l(v)$ to each node v
  - compute $l(\mathrm{LCA}(x,y))$ from $l(x)$ and $l(y)$

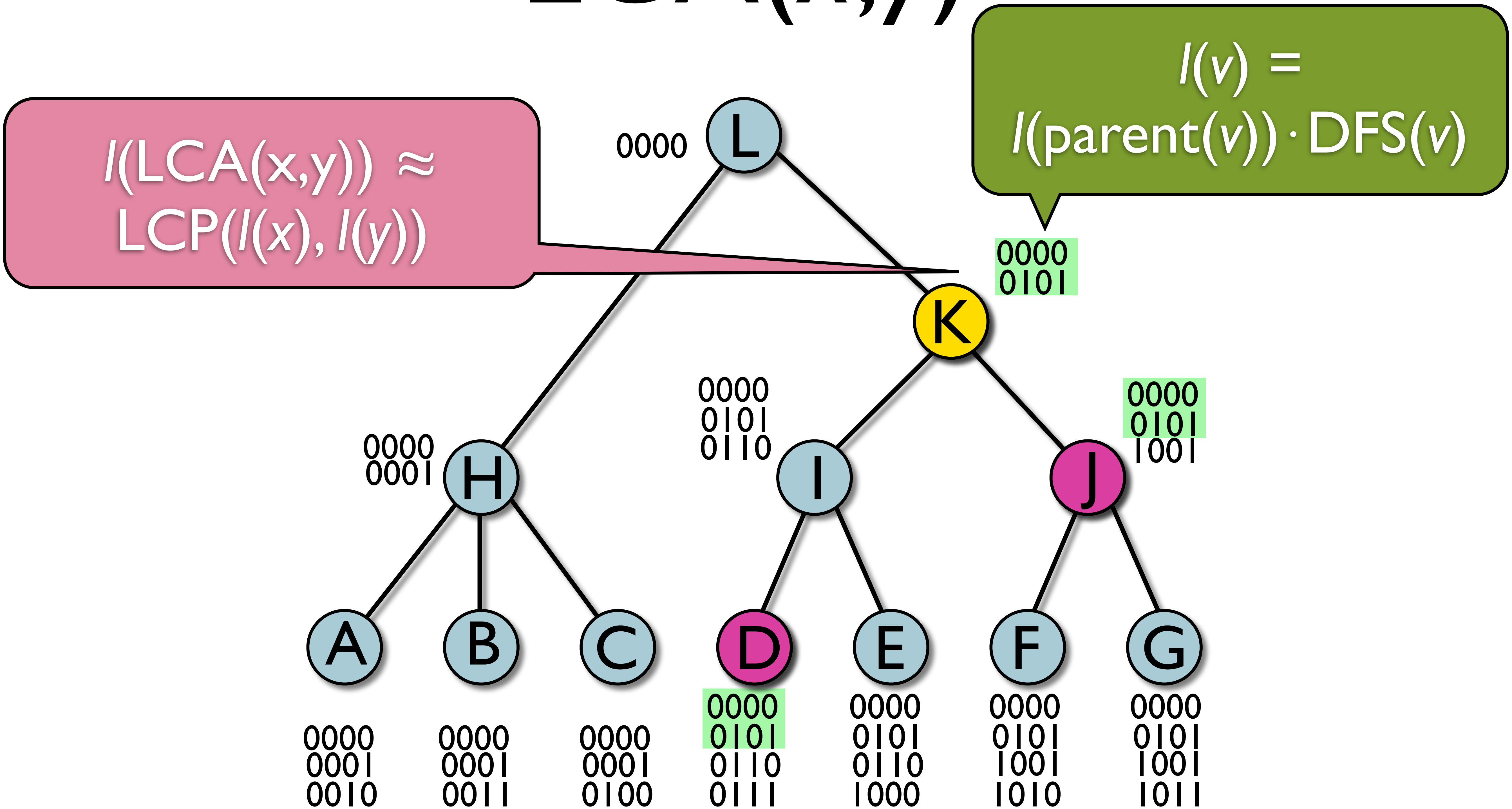
- goal:
  - short labels
  - fast query time

Simple Tree Labelings: parent(x,y)

parent(x,y) iff
first lg n bits of $l(x)$
= 2nd lg n bits of $l(y)$



Simple Tree Labelings: LCA(x,y)



Simple LCA-Labeling

- longest label length:
  - between $O(\lg^2 n)$ and $O(n \lg n)$ bits
  - $\Rightarrow$ cannot even compute LCP in $O(1)$ time
- in the following:
  - label length $O(\lg n)$ bits
  - $O(1)$ query time

Definitions

- node v:
  - $p(v)$ = parent of v
  - $c(v)$ = set of v's children
  - $size(v)$= #nodes in v's subtree T_v
- **heavy** nodes:
  - having largest subtree among its siblings
  - u heavy if $size(u) = \max\{size(w) : w \in c(p(u))\}$
  - take arbitrary child if max not unique
- all other nodes: **light** (incl. root)

Heavy Paths

- heavy nodes divide *T* into **heavy paths**:
  - from light node follow heavy nodes
  - continue recursively
  - **heavy path decomposition**
- $\langle v_1, v_2, \ldots, v_k \rangle$ heavy path
  - $v_1 = a(v_i)$ is the **apex** of v_i for all i
- **light size** of v:
  - $lsize(v) = size(v) - size(w)$ if w is v's heavy child

Labels

- **heavy label** $hl(v)$
  - to any node v
  - different for two nodes on one heavy path
  - can determine if $i < j$ from v_i, v_j on $\langle v_1, v_2, \ldots, v_k \rangle$
- **light label** ll(v):
  - only to light nodes v
  - different for nodes with same parent
- **label** $l(\mathrm{v}) = l(p(a(v))) \cdot ll(a(v)) \cdot hl(v)$

Answering LCA(x,y)

- compute LCP of $l(x)$ and $l(y)$
- 2 cases
  - depending on whether **mismatch** occurs in hl or ll
  - need **helper label** (0 ≙ hl, 1 ≙ ll)
- see blackboard

Analysis: Idea

- $hl(v)$ **repeated** in all nodes below v apart from those below heavy child

  $\Rightarrow$ $hl(v)$ occurs $lsize(v)$ times

  $\leadsto$ use **shorter** heavy labels for **large lsizes**

- $ll(v)$ occurs in all nodes below v

  $\Rightarrow$ $ll(v)$ occurs $size(v)$ times

  $\leadsto$ use **shorter** light labels for **large subtrees**

Precise Analysis

- see blackboard